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A GENERAL BOUNDARY INTEGRAL FORMULATION FOR THE NUMERICAL SOLUTION OF PLATE BENDING PROBLEMS†

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Abstract—A direct approach is employed to obtain a general formulation of plate bending problems in terms of a pair of singular integral equations involving displacement, normal slope, bending moment and shear on the plate boundary. These equations are coupled with prescribed boundary conditions involving these same variables furnishing a convenient basis for numerical solution. A simple discretization scheme is described and two model problems treated to illustrate the nature and quality of some typical results.

INTRODUCTION

The formulation of elastic plate bending problems via boundary integral equations furnishes the basis for an alternative to finite difference and finite element approaches to the numerical solution of such problems. This was recognized by Massonnet[1] who suggested a formulation suitable for clamped plates by analogy with the problem of determining an Airy stress function for a particular plane elastostatic problem. Jaswon and Maiti[2] produced a boundary integral treatment for uniformly loaded clamped or simply supported plates in terms of two source distribution densities generating harmonic potentials which are then related to the plate displacement. Other authors (e.g. [3, 4]) have proposed methods suitable for particular, but restricted, classes of plate bending problems admitting only special geometries or specific classes of boundary conditions.

More recently Altiero and Sikarski[5] have suggested a more general treatment by imbedding the problem in one for which the Green's function for a concentrated load and moment are known. For example, by considering a clamped circular plate large enough to contain the original plate region, an unspecified line load of concentrated force and moment may be placed along the boundary of the original plate region, and the solution throughout the plate represented in terms of a "boundary integral" involving these unknown densities and the well known Green's functions for a clamped circular plate. Prescribed boundary conditions on the original plate boundary may then be used to generate a pair of coupled integral equations for the force and moment densities. The authors, however, acknowledge serious practical difficulties with other than clamped boundary conditions and hence treat only this case in their paper.

In the present work, a more direct approach is employed to obtain a general formulation in terms of a pair of integral equations involving displacement, normal slope, bending moment, and equivalent shear on the boundary, which are coupled with prescribed boundary conditions involving these same variables.‡ A simple discretization scheme is then described and two model problems treated.

PRELIMINARIES

Some of the notation we shall use is indicated in Fig. 1, which shows a portion of the plate. Specifically, the plate is modelled by a bounded plane region Ω with total boundary $\partial\Omega$ that is piecewise twice continuously differentiable, i.e. the curvature is bounded and continuous except

†The major portion of this work was completed while the author was on leave at the California Institute of Technology, September 1977 to January 1978.

‡After this manuscript was prepared the author was made aware of work along these same lines by G. Bezine, Université de Poitiers. Also, the work of Forbes and Robinson, Numerical analysis of elastic plates and shallow shells by an integral equation method, University of Illinois Structural Research Series Report 345 (1969) is based on a similar approach.

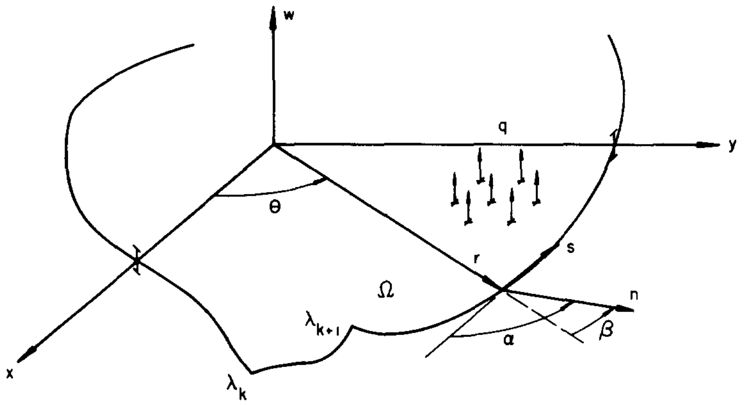

Fig. 1. Relevant notation.

possibly at the (finite number of) corner points λ_k, $k = 1, \ldots, K$; and finally we require that each corner point have a non-zero interior angle. We suppose the plate to be loaded by a transverse load intensity q defined on Ω, and note that for a uniform linearly elastic plate the deflection w is governed by the differential equation

$$\nabla^4 w = q/D \text{ in } \quad \Omega \tag{1}$$

with suitable boundary conditions, left unspecified at present, imposed on $\partial\Omega$. The plate stiffness is denoted $D = Eh^3/12(1-\nu^2)$ where E is the elastic modulus, h is the plate thickness and ν is Poisson's ratio, and ∇^4 is the iterated Laplacian operator.

Now let u be the deflection associated with some other particular loading on Ω and suitable boundary conditions on $\partial\Omega$, and consider the symmetric bilinear form

$$\mathcal{U}(w, u) = D\int_\Omega \left\{\nabla^2 w \nabla^2 u - (1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 u}{\partial x \partial y}\right)\right\} \mathrm{d}a \tag{2}$$

where the derivatives are referred to any convenient cartesian coordinate system in the plane. We not that $\mathcal{U}(w, w)$ is precisely twice the strain energy of the plate subjected to the displacement field $w = w(x, y)$.

If eqn (2) is formally integrated by parts twice in u, and the resulting tangential derivative on the boundary integrated between corners, we obtain the formula

$$\mathcal{U}(w, u) = D\int_\Omega u\nabla^4 w \, \mathrm{d}a + \int_{\partial\Omega}\left\{\mathcal{V}_n(w)u - \mathcal{M}_n(w)\frac{\mathrm{d}u}{\mathrm{d}n}\right\}\mathrm{d}s + \sum_{k=1}^{K} [[\![\mathcal{M}_t(w)]\!]u]_{\lambda_k} \tag{3}$$

where, at a regular point of the boundary

$$\begin{aligned}
\mathcal{M}_n(w) &= D\left\{-\nabla^2 w + (1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\sin^2\alpha + \frac{\partial^2 w}{\partial y^2}\cos^2\alpha - 2\frac{\partial^2 w}{\partial x \partial y}\sin\alpha\cos\alpha\right]\right\} \\
&= D/2\left\{-(1+\nu)\nabla^2 w + (1-\nu)\left[\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right)\cos 2\alpha - 2\frac{\partial^2 w}{\partial x \partial y}\sin 2\alpha\right]\right\} \\
\mathcal{M}_t(w) &= -D(1-\nu)\left\{\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right)\sin\alpha\cos\alpha + \frac{\partial^2 w}{\partial x \partial y}(\cos^2\alpha - \sin^2\alpha)\right\} \\
&= -\frac{D(1-\nu)}{2}\left\{\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right)\sin 2\alpha + 2\frac{\partial^2 w}{\partial x \partial y}\cos 2\alpha\right\} \\
\mathcal{V}_n(w) &= -D\frac{\mathrm{d}}{\mathrm{d}n}\nabla^2 w + \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}_t(w)
\end{aligned} \tag{4}$$

with α the angle from the x-axis to the outer normal and $\mathrm{d}/\mathrm{d}n$, $\mathrm{d}/\mathrm{d}s$ denoting the normal and tangential derivatives on $\partial\Omega$. Finally, $[\![\cdot]\!]$ denotes the discontinuity jump in the direction of increasing arc length and the summation extends over all the corner points of $\partial\Omega$. Upon invoking the symmetry of $\mathcal{U}(\cdot,\cdot)$ we obtain the formal integral identity

$$\int_{\partial\Omega}\left\{\mathcal{V}_n(u)w-\mathcal{M}_n(u)\frac{\mathrm{d}w}{\mathrm{d}n}+\frac{\mathrm{d}u}{\mathrm{d}n}\mathcal{M}_n(w)-u\mathcal{V}_n(w)\right\}\mathrm{d}s=D\int_{\Omega}(u\nabla^4w-w\nabla^4u)\,\mathrm{d}a$$
$$+\sum_{k=1}^{K}\Big[[\![\mathcal{M}_t(w)]\!]u-[\![\mathcal{M}_t(u)]\!]w\Big]_{\lambda_k}. \quad (5)$$

The boundary operators $\mathcal{M}_n(\cdot)$, $\mathcal{M}_t(\cdot)$, $\mathcal{V}_n(\cdot)$ produce, respectively, the bending moment, twisting moment and equivalent shear. Upon extending these operators to regular arcs in the interior of the plate, the usual definition of stress and couple results on coordinate lines as indicated in Fig. 2 is then consistent with constitutive equations of the form

$$\begin{aligned}
M_{xx}&=\mathcal{M}_n(w)|_{\alpha=0,\pi}=-D\left\{\frac{\partial^2w}{\partial x^2}+\nu\frac{\partial^2w}{\partial y^2}\right\}\\
M_{yy}&=\mathcal{M}_n(w)|_{\alpha=\pm\pi/2}=-D\left\{\frac{\partial^2w}{\partial y^2}+\nu\frac{\partial^2w}{\partial x^2}\right\}\\
M_{xy}&=\mathcal{M}_t(w)|_{\alpha=0,\pi}=-\mathcal{M}_t(w)|_{\alpha=\pm\pi/2}=-D(1-\nu)\frac{\partial^2w}{\partial x\partial y}\\
Q_x&=Q_n(w)|_{\alpha=0}=-Q_n(w)|_{\alpha=\pi}=-D\frac{\partial}{\partial x}\nabla^2w \qquad (6)\\
Q_y&=Q_n(w)|_{\alpha=\pi/2}=-Q_n(w)|_{\alpha=-\pi/2}=-D\frac{\partial}{\partial y}\nabla^2w
\end{aligned}$$

where

$$Q_n(w)=-D\frac{\mathrm{d}}{\mathrm{d}n}\nabla^2w=\mathcal{V}_n(w)-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}_t(w). \quad (7)$$

It will prove convenient for later calculations to refer the definitions in eqn (4) to a polar coordinate system. Thus, if we denote by β the angle from the radial direction to the outer normal, upon regarding w as a function of the polar coordinates (r,θ) and noting that $\beta=\alpha-\theta$, we find

$$\mathcal{M}_n(w)=D/2\left\{-(1+\nu)\nabla^2w+(1-\nu)\left[\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2w}{\partial\theta^2}-\frac{\partial^2w}{\partial r^2}\right)\cos 2\beta-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\sin 2\beta\right]\right\}$$
$$\mathcal{M}_t(w)=-\frac{D(1-\nu)}{2}\left\{\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2w}{\partial\theta^2}-\frac{\partial^2w}{\partial r^2}\right)\sin 2\beta+2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\cos 2\beta\right\}. \quad (8)$$

Since $\mathcal{M}_t(w)$ depends on β (or α), as well as r and θ (or x and y), the tangential derivative contains additional chain rule terms. We denote by $1/R$ the curvature at a regular boundary

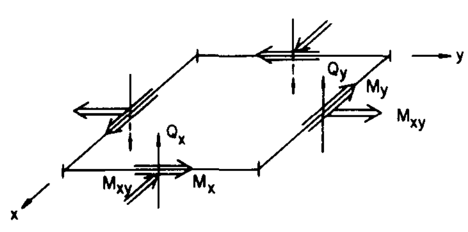

Fig. 2. Cartesian stress and couple resultants.

point (adopting the sign convention that a negative curvature indicates the center of curvature is on the outward normal); then clearly

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = \frac{1}{R}, \qquad \frac{\mathrm{d}\beta}{\mathrm{d}s} = \frac{1}{R} - \frac{\cos\beta}{r}.$$

Consequently, considering $\mathcal{M}_t(w)$ alternatively as a function of (x, y, α) or of (r, θ, β) we find

$$\begin{aligned}
\mathscr{V}_n(w) &= \left[-D\frac{\partial}{\partial x}\nabla^2 w + \frac{\partial}{\partial y}\mathcal{M}_t(w)\right]\cos\alpha \\
&\quad - \left[D\frac{\partial}{\partial y}\nabla^2 w + \frac{\partial}{\partial x}\mathcal{M}_t(w)\right]\sin\alpha + \frac{1}{R}\frac{\partial}{\partial\alpha}\mathcal{M}_t(w) \\
&= \left[-D\frac{\partial}{\partial r}\nabla^2 w + \frac{1}{r}\left(\frac{\partial}{\partial\theta}\mathcal{M}_t(w) - \frac{\partial}{\partial\beta}\mathcal{M}_t(w)\right)\right]\cos\beta \\
&\quad - \left[\frac{D}{r}\frac{\partial}{\partial\theta}\nabla^2 w + \frac{\partial}{\partial r}\mathcal{M}_t(w)\right]\sin\beta + \frac{1}{R}\frac{\partial}{\partial\beta}\mathcal{M}_t(w).
\end{aligned} \tag{9}$$

Now suppose that the boundary value problem posed by eqn (1), with suitable boundary conditions appended, has a reasonably well behaved solution. In particular, we require that the deflection and derivatives up to third order be continuous and bounded† on Ω. Let P be an interior point of Ω and suppose $d > 0$ is its distance from $\partial\Omega$. Place the origin of polar coordinates at P and delete from Ω a circular region of radius $\rho < d$. Then in the resulting region Ω_ρ we introduce the following displacement field (corresponding to a concentrated force at P) which we note is continuously differentiable as often as desired on Ω_ρ:

$$u = W(r) = \frac{1}{8\pi D} r^2 \ln r \tag{10}$$

and with the aid of eqns (8) and (9), on $\partial\Omega_\rho$ we find

$$\begin{aligned}
N(r, \theta, \beta) &= \frac{\mathrm{d}W}{\mathrm{d}n} = \frac{1}{8\pi D}(1 + 2\ln r) r\cos\beta \\
M(r, \theta, \beta) &= \mathcal{M}_n(W) = -\frac{1+\nu}{4\pi}(1 + \ln r) - \frac{1-\nu}{8\pi}\cos 2\beta \\
T(r, \theta, \beta) &= \mathcal{M}_t(W) = \frac{1-\nu}{8\pi}\sin 2\beta \\
V(r, \theta, \beta) &= \mathscr{V}_n(W) = -\frac{\cos\beta}{4\pi r}[2 + (1-\nu)\cos 2\beta] + \frac{1-\nu}{4\pi R}\cos 2\beta
\end{aligned} \tag{11}$$

and

$$\nabla^4 W = 0.$$

Clearly, eqn (5) holds on Ω_ρ for every $0 < \rho < \mathrm{d}$ where now $\partial\Omega_\rho$ contains, in addition to $\partial\Omega$, the circular boundary surrounding P. The contribution from this part of the boundary, for vanishingly small ρ, is just the deflection w evaluated at P, consequently we deduce the "Green's formula"

$$\begin{aligned}
w|_{P\in\Omega} = -\int_{\partial\Omega}\left\{Vw - M\frac{\mathrm{d}w}{\mathrm{d}n} + N\mathcal{M}_n(w) - W\mathscr{V}_n(w)\right\}\mathrm{d}s + D\int_\Omega W\nabla^4 w\,\mathrm{d}a \\
+ \sum_{k=1}^{K}\{[\![\mathcal{M}_t(w)]\!]W - [\![T]\!]w\}_{\lambda_k}.
\end{aligned} \tag{12}$$

†This condition could be violated in problems of interest, for example, under some circumstances in the neighborhood of corners of the plate boundary the moment and shear could become unbounded. A technique for calculating and removing such singularities will be sketched later.

Furthermore, only the functions V, M, N, W and T on the right-hand side of eqn (12) depend on the location of P and these are given explicitly (in terms of r, θ, β) by eqns (10) and (11). We note that directional derivatives at P in the direction making an angle ψ with the x-axis (i.e. $\theta = 0$) may be computed by formal chain rule differentiation upon noting

$$\frac{\mathrm{d}r}{\mathrm{d}s_\psi} = \cos(\theta - \psi), \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s_\psi} = -\frac{\mathrm{d}\beta}{\mathrm{d}s_\psi} = \frac{\sin(\theta - \psi)}{r}. \tag{13}$$

Formulas for the derivatives of w to any order in the interior follow by repeated differentiation of eqn (12) provided the integral involving $D\nabla^4 w (= q)$ continues to converge, and this will certainly be the case through at least third derivatives. Thus, eqn (12) "solves" the problem in terms of the boundary values of the deflection w, the normal derivative $\mathrm{d}w/\mathrm{d}n$, the bending moment $\mathcal{M}_n(w)$, and the equivalent shear $\mathcal{V}_n(w)$, along with the corner jumps in the twisting moment $[\![\mathcal{M}_t(w)]\!]_{\lambda_k}$, $k = 1, \ldots, K$. In the next section we derive a pair of coupled integral equations which, together with prescribed boundary conditions, determine these boundary functions.

PLATE BOUNDARY INTEGRAL EQUATIONS

We now place the origin point P on the boundary $\partial\Omega$, and for generality suppose that P is a corner point with internal angle $\kappa\pi$; should P be a regular point of $\partial\Omega$ (at which the tangent turns continuously and the boundary data is smooth) we merely set $\kappa = 1$ and note the continuity of relevant quantities. Again, we suppose that the posed boundary value problem has a solution with continuous bounded derivatives up to third order in Ω. The kernel functions we will introduce will be continuous everywhere except possibly at P, hence we again form the region Ω_ρ by deleting from Ω those points within a distance ρ from P. For ρ small enough the boundary $\partial\Omega_\rho$ is changed from $\partial\Omega$ by the introduction of a new arc C_ρ and two new corner points λ^+ and λ^-, while deleting the arcs denoted C_ρ^+ and C_ρ^- and (possibly) the corner at P. This is illustrated in Fig. 3. The remaining boundary that Ω and Ω_ρ have in common is denoted C_ρ^*. A local polar coordinate system is also installed at P as shown.

The identity (5) applied to the region Ω_ρ for biharmonic u and vanishingly small ρ, on assuming the indicated limits exits, takes the form

$$\mathcal{J}_u + \int_{C^*} \left\{\mathcal{V}_n(u)w - \mathcal{M}_n(u)\frac{\mathrm{d}w}{\mathrm{d}n} + \frac{\mathrm{d}u}{\mathrm{d}n}\mathcal{M}_n(w) - u\mathcal{V}_n(w)\right\}\mathrm{d}s + \sum_{k=1}^{K}{}^{*}\,[w[\![\mathcal{M}_t(u)]\!] - u[\![\mathcal{M}_t(w)]\!]]_{\lambda_k}$$
$$+ [\![w[\![\mathcal{M}_t(u)]\!] - u[\![\mathcal{M}_t(w)]\!]]\!]_P = \int_\Omega qu\,\mathrm{d}a \tag{14}$$

where

$$\mathcal{J}_u = \lim_{\rho \to 0} \int_{C_\rho} \left\{\mathcal{V}_n(u)w - \mathcal{M}_n(u)\frac{\mathrm{d}w}{\mathrm{d}n} + \frac{\mathrm{d}u}{\mathrm{d}n}\mathcal{M}_n(w) - u\mathcal{V}_n(w)\right\}\mathrm{d}s \tag{15}$$

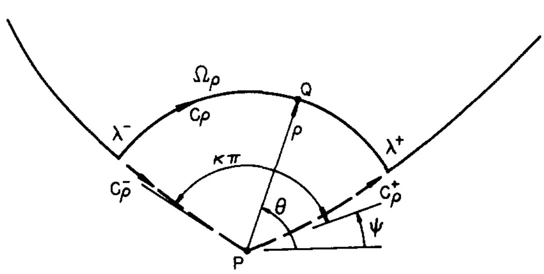

Fig. 3. Region and boundary changes from Ω to Ω_ρ.

is evaluated in a neighborhood of the boundary point P, and we have introduced the notation

$$\int_{C^*} (\cdot)\,\mathrm{d}s = \lim_{\rho\to 0} \int_{C^*_\rho} (\cdot)\,\mathrm{d}s \tag{16}$$

for the Cauchy Principal Value of the boundary integral. Also, if P is indeed a corner point its contribution is to be deleted in forming the sum $\sum_{k=1}^{K}{}^{*}(\cdot)_{\lambda_k}$, and finally

$$[\![\cdot]\!]_P = \lim_{\rho\to 0} \{[\cdot]_{\lambda^+} + [\cdot]_{\lambda^-}\}. \tag{17}$$

For future reference in evaluating these limits we make the following observations: At any interior point Q of C_ρ,

$$\beta = \pi, \qquad \frac{1}{R} = -\frac{1}{\rho}$$

$$w|_Q - w|_P = -\rho \left.\frac{\mathrm{d}w}{\mathrm{d}n}\right|_Q + \mathcal{O}(\rho^2). \tag{18}$$

Also,

$$\theta|_{\lambda^+} = \psi + \frac{\rho}{2R^+} + \mathcal{O}(\rho^2), \qquad \theta|_{\lambda^-} = \psi + \kappa\pi - \frac{\rho}{2R^-} + \mathcal{O}(\rho^2) \tag{19}$$

where $1/R^+$, $1/R^-$ are the limiting values of the curvature on either side of P. Furthermore, on C_ρ^+ and C_ρ^- we have

$$\beta = \mp\left(\frac{\pi}{2} - \frac{r}{2R^\pm}\right) + \mathcal{O}(r^2) \tag{20}$$

so that on these arcs

$$\cos\beta = \frac{r}{2R^\pm} + \mathcal{O}(r^2) \qquad \sin\beta = \mp 1 + \mathcal{O}(r^2)$$

$$\cos 2\beta = -1 + \mathcal{O}(r^2) \qquad \sin 2\beta = \mp\frac{r}{R^\pm} + \mathcal{O}(r^2). \tag{21}$$

Now in eqn (14) consider the special choice for u defined by eqns (10) and (11). It immediately follows from the assumed smoothness of w and the observation (18) that the limiting value of eqn (15) is just

$$\mathcal{J}_W = 1/2\kappa w|_P \tag{22}$$

and we recall that if P is a regular point of $\partial\Omega$, then $\kappa = 1$. The convergence of the ingral over C^* is easily shown; indeed, for ρ fixed and $0<\epsilon<r<\rho$ we have, on C_ρ^+ and C_ρ^-,

$$V = -\frac{3-\nu}{8\pi R^\pm} + \mathcal{O}(r^2)$$

$$M = -\frac{1+\nu}{4\pi}\ln r - \frac{1+3\nu}{8\pi} + \mathcal{O}(r^2)$$

$$N = \mathcal{O}(r^2 \ln r) = o(r)$$

$$W = \mathcal{O}(r^2 \ln r) = o(r). \tag{23}$$

Consequently, the contribution to the limiting value of the boundary integral from the arcs C_ρ^+ and C_ρ^- may be estimated by

$$\lim_{\epsilon\to 0}\int_\epsilon^\rho \left\{-\frac{3-\nu}{8\pi}\left[\frac{1}{R^-}w|_{C_\rho^-}+\frac{1}{R^+}w|_{C_\rho^+}\right]+\frac{1+\nu}{8\pi}\left[2\ln r+\frac{1+3\nu}{1+\nu}\right]\left[\frac{\partial w}{\partial n}\bigg|_{C_\rho^-}+\frac{\partial w}{\partial n}\bigg|_{C_\rho^+}\right]+\mathcal{O}(r)\right\}\mathrm{d}r$$
$$=-\frac{(1-\nu)\rho}{2\pi R_P}w|_P+\frac{1+\nu}{4\pi}\rho\left(2\ln\rho+\frac{1+3\nu}{1+\nu}\right)\frac{\mathrm{d}w}{\mathrm{d}n}\bigg|_P+o(\rho) \tag{24}$$

where

$$R_P=\frac{2R^+R^-}{R^++R^-},\qquad \frac{\mathrm{d}w}{\mathrm{d}n}\bigg|_P=\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}n}\bigg|_{P^-}+\frac{\mathrm{d}w}{\mathrm{d}n}\bigg|_{P^+}\right). \tag{25}$$

Finally, it is easy to see that

$$\lim_{\rho\to 0}[\![T]\!]_{\lambda^+}=\lim_{\rho\to 0}[\![T]\!]_{\lambda^-}=\lim_{r\to 0}W(r,\theta)=0.$$

Hence, the contribution from the corner terms at λ^+ and λ^- vanishes, and we may write eqn (14) as

$$1/2\kappa w|_P+\int_{C^*}\left\{Vw-M\frac{\mathrm{d}w}{\mathrm{d}n}+N\mathcal{M}_n(w)-W\mathcal{V}_n(w)\right\}\mathrm{d}s$$
$$+\sum_{k=1}^{K}{}^*\{w[\![T]\!]-W[\![\mathcal{M}_t(w)]\!]\}_{\lambda_k}=\int_\Omega qW\,\mathrm{d}a. \tag{26}$$

We next obtain a representation formula for the derivatives of w at P. To this end we introduce a local $\epsilon\eta$-coordinate system rotated an angle $\psi+\gamma$ from the $\theta=0$ direction and introduce the new polar angle $\phi=\theta-(\psi+\gamma)$ as illustrated in Fig. 4. Then an appropriate biharhomic fundamental solution is

$$u=W_\gamma(r,\phi)=\frac{1}{2\pi D}r\ln r\cos\phi \tag{27}$$

and from eqns (8) and (9) we compute

$$N_\gamma(r,\phi,\beta)=\frac{\mathrm{d}W_\gamma}{\mathrm{d}n}=\frac{1}{2\pi D}\{\cos\phi\cos\beta+\ln r\cos(\phi+\beta)\}$$

$$M_\gamma(r,\phi,\beta)=\mathcal{M}_n(W_\gamma)=-\frac{1+\nu}{2\pi}\frac{\cos\phi}{r}+\frac{1-\nu}{2\pi}\frac{\sin\phi}{r}\sin 2\beta$$

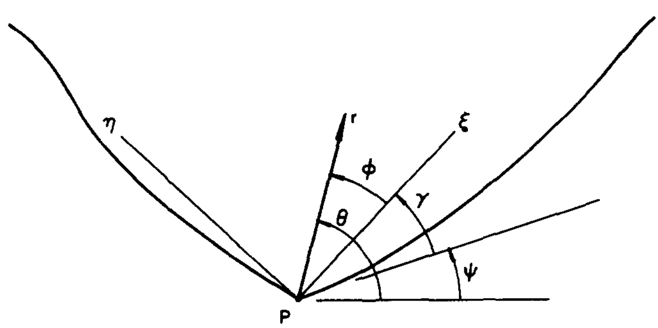

Fig. 4. Notation for origin at a plate corner.

$$V_\gamma(r,\phi,\beta)=\mathscr{V}_n(W_\gamma)=\frac{1}{2\pi r^2}\{\cos(\beta-\phi)[2+(1-\nu)\cos 2\beta]$$

$$+2(1-\nu)\sin\phi\cos\beta\sin 2\beta\}-\frac{1-\nu}{\pi Rr}\sin\phi\sin 2\beta$$

$$T_\gamma(r,\phi,\beta)=\mathscr{M}_t(W_\gamma)=\frac{1-\nu}{2\pi}\frac{\sin\phi}{r}\cos 2\beta. \tag{28}$$

In this case, using the observations (18) we not that on C_ρ

$$V_\gamma=-\frac{3-\nu}{2\pi}\frac{\cos\phi}{\rho^2}$$

$$M_\gamma=-\frac{1+\nu}{2\pi}\frac{\cos\phi}{\rho}$$

$$N_\gamma=-\frac{1}{2\pi D}(1+\ln\rho)\cos\phi$$

$$W_\gamma=\frac{1}{2\pi D}\rho\ln\rho\cos\phi \tag{29}$$

and in general $\mathscr{J}_{W_\gamma}$ does not converge. We may remedy the situation, however, by noting that if we replace w in eqn (14) with $\hat{w}\equiv w-w|_P$, all derivatives remain unaltered, and with the aid of the estimates $(18)_3$ and (19) we compute

$$\begin{aligned}\hat{\mathscr{J}}_{W_\gamma}&=\lim_{\rho\to0}\int_{C_\rho}\left\{V_\gamma(w-w|_P)-M_\gamma\frac{dw}{dn}+N_\gamma\mathscr{M}_n(w)-W_\gamma\mathscr{V}_n(w)\right\}ds\\&=-\int_{-\gamma}^{\kappa\pi-\gamma}\frac{2}{\pi}\cos\phi\left[\frac{\partial w}{\partial\xi}\bigg|_P\cos\phi+\frac{\partial w}{\partial\eta}\bigg|_P\sin\phi\right]d\phi\\&=-\left\{\kappa+\frac{1}{2\pi}[\sin 2\gamma+\sin 2(\kappa\pi-\gamma)]\right\}\frac{dw}{d\xi}\bigg|_P\\&\quad-\frac{1}{2\pi}[\cos 2\gamma-\cos 2(\kappa\pi-\gamma)]\frac{dw}{d\eta}\bigg|_P.\end{aligned} \tag{30}$$

To establish the convergence of the integral over C^* we again estimate the contribution from C_ρ^+ and C_ρ^-. Introducing the notation

$$\phi^+=-\gamma,\qquad \phi^-=\kappa\pi-\gamma \tag{31}$$

we have the following estimates on C_ρ^+ and C_ρ^-:

$$\hat{w}=w-w|_P=r\left[\frac{\partial w}{\partial\xi}\bigg|_P\cos\phi^\pm+\frac{\partial w}{\partial\eta}\bigg|_P\sin\phi^\pm\right]+\mathcal{O}(r^2)$$

$$\frac{dw}{dn}=\mp\left\{\frac{\partial w}{\partial\xi}\bigg|_P\sin\phi^\pm-\frac{\partial w}{\partial\eta}\bigg|_P\cos\phi^\pm\right\}+\mathcal{O}(r)$$

$$V_\gamma=\mp\frac{1+\nu}{2\pi r^2}\sin\phi^\pm+\mathcal{O}(r^2)$$

$$M_\gamma=-\frac{1+\nu}{2\pi r}\cos\phi^\pm+\mathcal{O}(1)$$

$$N_\gamma=\pm\frac{1nr}{2\pi D}\sin\phi^\pm+o(1)$$

$$W_\gamma=o(1). \tag{32}$$

Then paralleling the computation of eqn (24) we see that for fixed ρ, the contribution to the boundary integral from C_ρ^+ and C_ρ^- is estimated by

$$\lim_{\epsilon\to 0}\int_\epsilon^\rho \left\{\frac{1+\nu}{2\pi r^2}\left(-\sin\phi^+ r\left[\frac{\partial w}{\partial \xi}\bigg|_P \cos\phi^+ + r\frac{\partial w}{\partial \eta}\bigg|_P \sin\phi^+\right]\right.\right.$$

$$\left. + \sin\phi^- r\left[\frac{\partial w}{\partial \xi}\bigg|_P \cos\phi^- + r\frac{\partial w}{\partial \eta}\bigg|_P \sin\phi^-\right]\right)$$

$$-\frac{1+\nu}{2\pi r}\left(\cos\phi^+\left[-\frac{\partial w}{\partial \xi}\bigg|_P \sin\phi^+ + \frac{\partial w}{\partial \eta}\bigg|_P \cos\phi^+\right] + \cos\phi^-\left[\frac{\mathrm{d}w}{\mathrm{d}\xi}\bigg|_P \sin\phi^- - \frac{\partial w}{\partial \eta}\bigg|_P \cos\phi^-\right]\right)$$

$$\left. + \frac{\ln r}{2\pi D}[\![\sin\phi \mathscr{M}_n(w)]\!]_P + \mathcal{O}(1)\right\}\mathrm{d}r = \frac{\rho\ln\rho}{2\pi D}[\![\sin\phi\mathscr{M}_n(w)]\!]_P + \mathcal{O}(\rho) \tag{33}$$

and consequently the integral over C^* converges.

Finally, we note that

$$[\![\cos 2\beta]\!]_{\lambda^\pm} = \mp 2 + \mathcal{O}(\rho)$$

so that

$$[\![T]\!]_{\lambda^\pm} = \mp\frac{1-\nu}{\pi\rho}\sin\phi^\pm + \mathcal{O}(1). \tag{34}$$

Using eqn $(32)_1$ to estimate $w|_{\lambda^\pm}$ we compute a nonzero contribution from the corners at λ^+ and λ^-:

$$\{\hat{w}[\![T_\gamma]\!]\}_P = \frac{1-\nu}{\pi}\left\{-\sin\phi^+\left[\frac{\partial w}{\partial \xi}\bigg|_P \cos\phi^+ + \frac{\partial w}{\partial \eta}\bigg|_P \sin\phi^+\right]\right.$$

$$\left. + \sin\phi^-\left[\frac{\partial w}{\partial \xi}\bigg|_P \cos\phi^- + \frac{\partial w}{\partial \eta}\bigg|_P \sin\phi^-\right]\right\}$$

$$= \frac{1-\nu}{2\pi}[\sin 2\gamma + \sin 2(\kappa\pi-\gamma)]\frac{\partial w}{\partial \xi}\bigg|_P + \frac{1-\nu}{2\pi}[\cos 2\gamma - \cos 2(\kappa\pi - \gamma)]\frac{\partial w}{\partial \eta}\bigg|_P. \tag{35}$$

Combining this result with eqn (30) we obtain eqn (14) finally in the form

$$\kappa_\xi\frac{\partial w}{\partial \xi}\bigg|_P + \kappa_\eta\frac{\partial w}{\partial \eta}\bigg|_P + \int_{C^*}\left\{V_\gamma(w-w|_P) - M_\gamma\frac{\mathrm{d}w}{\mathrm{d}n} + N_\gamma\mathscr{M}_n(w) - W_\gamma\mathscr{V}_n(w)\right\}\mathrm{d}s$$

$$+\sum_{k=1}^{K}{}^*\{(w-w|_P)[\![T_\gamma]\!] - W_\gamma[\![\mathscr{M}_t(w)]\!]\}_{\lambda_k} = \int_\Omega qW_\gamma\,\mathrm{d}a \tag{36}$$

where

$$\kappa_\xi = -\kappa - \frac{\nu}{2\pi}[\sin 2\gamma + \sin 2(\kappa\pi - \gamma)]$$

$$\kappa_\eta = -\frac{\nu}{2\pi}[\cos 2\gamma - \cos 2(\kappa\pi-\gamma)]. \tag{37}$$

We note that if P is a regular point for which $\kappa = 1$, then $\kappa_\xi = -1$, $\kappa_\eta = 0$. Thus, if ξ is oriented along the outer normal, the first two terms in eqn (36) collapse to $-(\mathrm{d}w/\mathrm{d}n)|_P$. If $\kappa \neq 1$, for the particular choice $\gamma = \gamma_1 = (2+\kappa)\pi/2$ (i.e. the ξ_1-axis bisects the exterior angle) we find $\kappa_{\xi_1} = \kappa_1 = -\kappa - (\nu/\pi)\sin\kappa\pi$, $\kappa_\eta = 0$, whereas for $\gamma = \gamma_2 = (\kappa-1)(\pi/2)$ (the ξ_2-axis perpendicular to the corner angle bisector) we find $\kappa_{\xi_2} = \kappa_2 = -\kappa + (\nu/\pi)\sin\kappa\pi$, $\kappa_\eta = 0$. These two choices in eqn (36) yield directional derivatives in the ξ_1-and ξ_2-directions, consequently suitable linear combinations will give independent representations for the normal derivative to either side of the corner point P. Indeed, with the notation implied in Fig. 5 we can write

$$\kappa_1 \frac{\partial w}{\partial \xi_1}\bigg|_P = -\frac{\kappa + \frac{\nu}{\pi}\sin\kappa\pi}{2\sin\kappa\pi/2}\left\{\frac{dw}{dn}\bigg|_{P^+} + \frac{dw}{dn}\bigg|_{P^-}\right\}$$

$$\kappa_2 \frac{\partial w}{\partial \xi_2}\bigg|_P = -\frac{\kappa - \frac{\nu}{\pi}\sin\kappa\pi}{2\cos\kappa\pi/2}\left\{\frac{dw}{dn}\bigg|_{P^+} - \frac{dw}{dn}\bigg|_{P^-}\right\}. \tag{38}$$

We will return to this observation in the next section.

Equations (26) and (36), together with prescribed boundary data, should be sufficient to determine the remaining boundary data needed in eqn (12) to characterize the solution of the original problem. Efficient and accurate methods of discretizing and approximating the solution of these equations as well as a strategy for treating uniformly any of a wide class of different plate geometries, loadings, and boundary conditions, is currently under study. In the next section we outline a relatively unsophisticated approach and illustrate it with two simple examples.

NUMERICAL TREATMENT

The boundary is partitioned with a finite number of nodal points, care being taken to place a nodal point at each corner. The deflection w, normal slope dw/dn, bending moment $\mathcal{M}_n(w)$, and equivalent shear $\mathcal{V}_n(w)$ are to be defined on the boundary in terms of their nodal values.It should be noted that at corner nodes there are two distinct limiting values of dw/dn, $\mathcal{M}_n(w)$ and $\mathcal{V}_n(w)$, as well as the twisting moment jump $[\![\mathcal{M}_t(w)]\!]$ (the deflection is, of course, continuous and unaffected by the discontinuity in outer normal direction).

At each interior node we will need to determine the four variables, and four independent relations may be associated with each such node. These are two independent boundary conditions, and discretized versions of eqns (26) and (36) generated in the following manner. Over everv segment of boundary between nodes the variables are interpolated linearly and a convenient quadrature formula is used (in the following examples, a fifth order Gaussian rule). Evaluated in the quadratures are the coefficients of each nodal variable, and these are accumulated in a "stiffness matrix" along with terms outside the boundary integral.

It should be noted from eqns (24) and (33) so long as we use a quadrature formula which does not require a function evaluation at the origin node itself, no special treatment is required for the boundary segments adjacent to the singular point. Furthermore, because the variables in this formulation do not involve the boundary geometry explicitly, there is no need to approximate the shape of the boundary in discretizing the equations. This could be an important advantage of this particular formulation in problems involving curved boundaries.

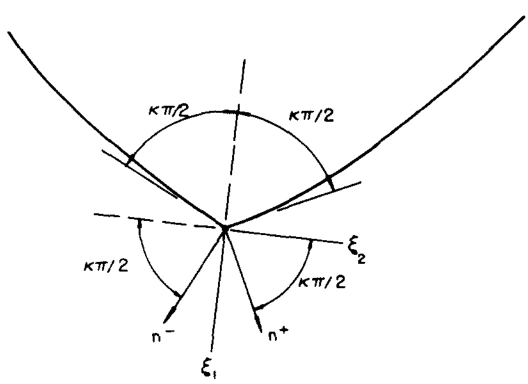

Fig. 5. Notation for directional derivatives and normal slopes at a corner.

At each corner node we require eight independent relations. One of these is obtained from eqn (26) and two others from eqn (36) and following remarks. The five additional relations needed at each corner node are furnished by boundary conditions, and if necessary, an examination of the asymptotic nature of the solution in a neighborhood of the corner. For example, at a corner of a rectangular plate, if both edges are free of support, then the right and left values of the bending moment and equivalent shear as well as the twisting moment jump must vanish. On the other hand, if both edges are simply supported, then boundary conditions dictate that the right and left values of bending moment as well as the deflection must vanish. The two additional conditions needed follow from an analysis of the asymptotic behavior of the solution in the neighborhood of a simply supported rectangular corner (as, for example in [6]) which dictates that the right and left values of the equivalent shear must vanish. The corner jump in the twisting moment, which is one of the remaining independent variables, may be interpreted as the concentrated reactive force at the corner required by classical plate theory. Finally, in the case where both edges at a rectangular corner are clamped, the right and left bending moment must vanish along with the right and left normal slope and the deflection.

It might be appropriate to remark here that an analysis of the asymptotic behavior of solutions with homogeneous boundary conditions at a corner point in the manner of [6] should also reveal all possible solutions which are consistent with bounded strain energy, but lead to cases where the smoothness assumptions laid down in the preceding section are violated. The significant variables associated with the corner node in such cases are the independent parameters of these singular eigensolutions. Adapting the procedure outlined in [7] (for evaluating stress intensity factors in plane problems) to the present case, the appropriate kernel functions for the boundary integral with origin at a singular corner are obtained from "complementary" eigensolutions, so that evaluation of the integral on the vanishingly small arc around the corner point leads to a linear combination of the desired parameters. Results obtained by incorporating this idea in the current scheme will be presented in a subsequent paper.

Finally, we can relax somewhat the requirement that the solution be smooth (through third derivatives) and allow concentrated force and moment loads in the plate interior. An analysis

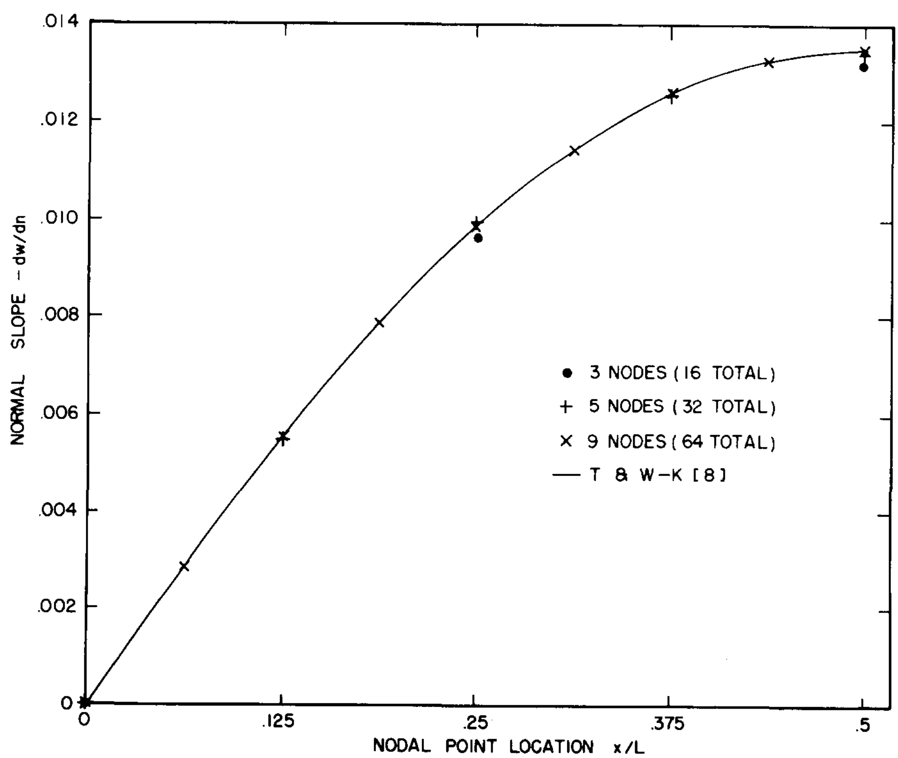

Fig. 6. Normal slope on half an edge of a uniformly-loaded simply-supported square plate.

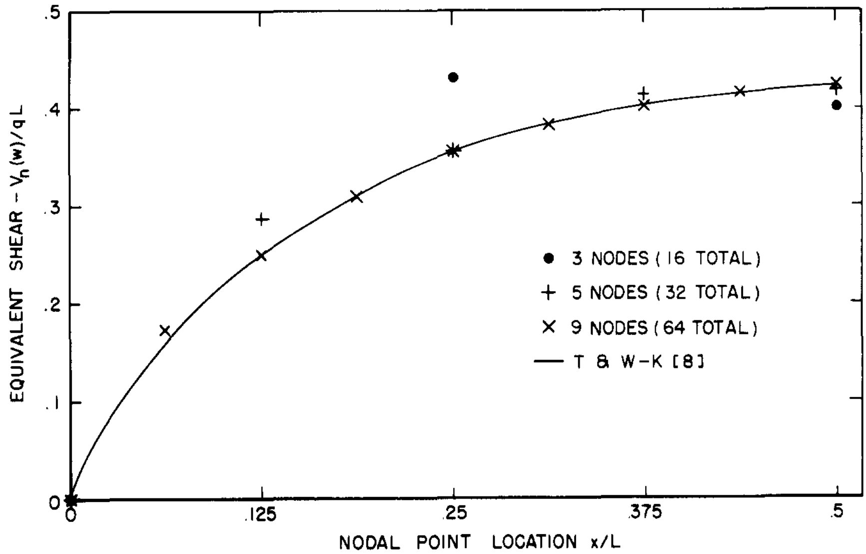

Fig. 7. Equivalent shear on half an edge of a uniformly-loaded simply-supported square plate.

similar to that leading to eqn (12) shows that all the integrals still converge, and the only modification required in eqns (26) and (36) is the addition of terms on the right-hand sides corresponding to the work done by the concentrated loads in the (known) auxiliary deflections W and W_γ.

For the present paper, numerical results were obtained for two model problems consisting of a square plate (side length L) subjected to a uniform load q and either hinged on all edges or clamped there. Numerical results were obtained for three meshes: 16, 32 or 64 equally spaced nodes on the entire boundary. Because of symmetry results are presented on only half of one edge involving 3, 5 or 9 nodes.

For the hinged plate, boundary conditions dictate that the displacement and bending moment vanish on the edges. Consequently, the nodal point variables of interest are the normal slope and equivalent shear as well as the corner reaction (corner jump in twisting moment). As is apparent in Figs. 6 and 7, convergence to the series solution given in [8] appears to be smooth

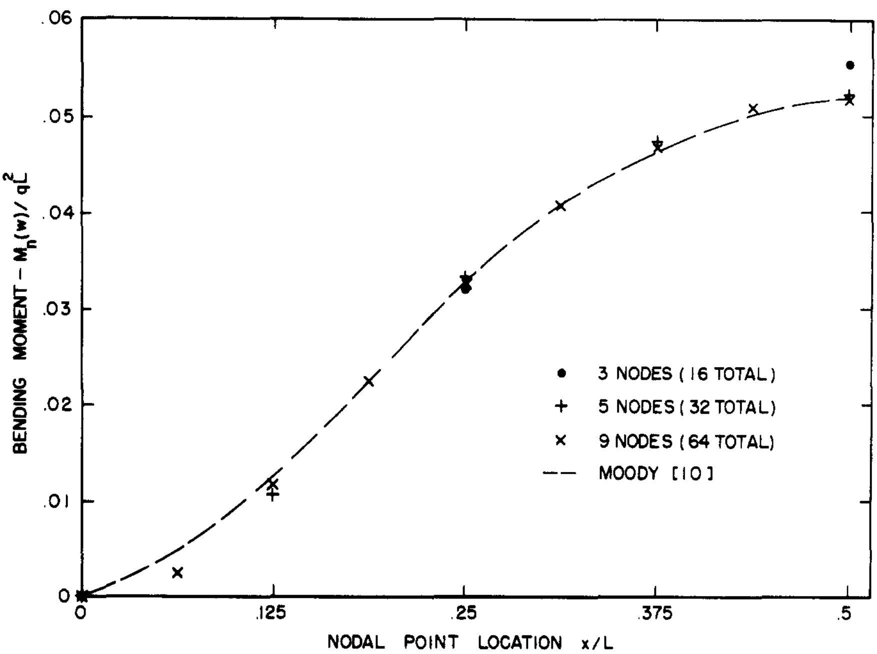

Fig. 8. Bending moment on half an edge of a uniformly-loaded clamped square plate.

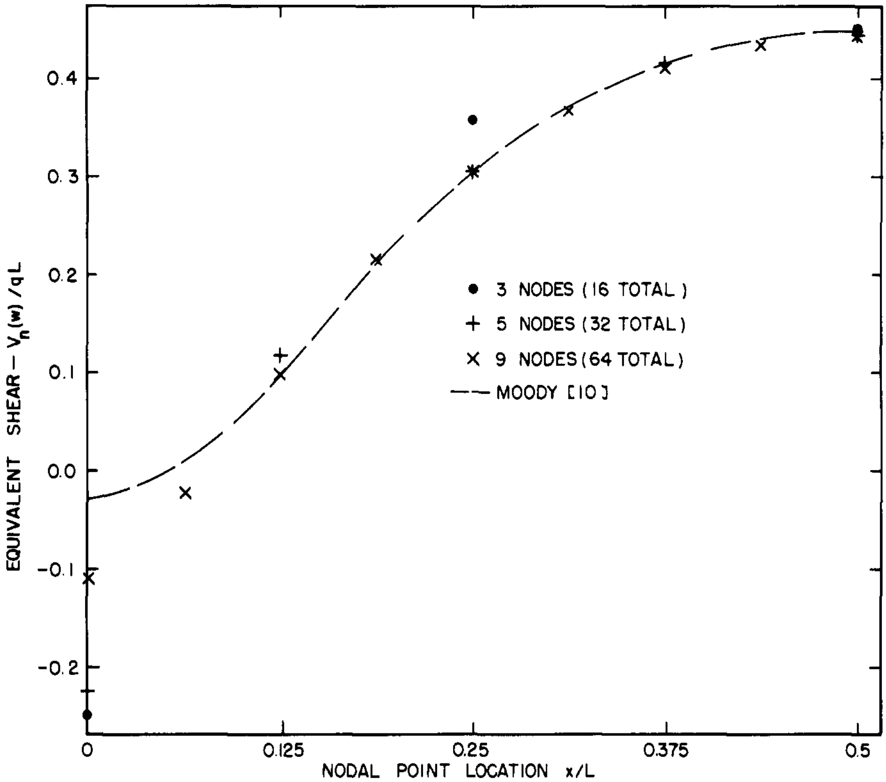

Fig. 9. Equivalent shear on half an edge of a uniformly-loaded clamped square plate.

and quite rapid. Furthermore, the independent calculation of the corner reaction is within 1.4% of the correct value for even the 16 nodal point mesh.

Bending moment and equivalent shear results for the clamped plate are shown in Figs. 8 and 9. The maximum values of the bending moment and shear at the midside node ($x/L = 0.5$) are with 1% of the values given by Wojtaszak[9] using 32 or 64 nodes. The comparison curves shown in Figs. 8 and 9 are interpolated from data given by Moody[10] for a uniformly loaded square plate clamped on three sides and hinged on the fourth. While the boundary conditions are different, the shear and moment on the edge opposite the hinge should be reasonably close to the case of all edges clamped and, indeed, the maximum values are within a few per cent of those quoted above.

Corner Reaction—$[\![\mathcal{M}_t(w)]\!]/qL^2$

	16 nodes	32 nodes	64 nodes	exact
Hinged Plate	0.0641	0.0651	0.0648	0.0650
Clamped Plate	0.0260	0.0128	0.0037	0

REFERENCES

1. C. E. Massonnet, Numerical use of integral procedures. *Stress Analysis* (Edited by O. C. Zienkiewicz and G. S. Holister), pp. 198–235. Wiley, London (1965).
2. M. A. Jaswon and M. Maiti, An integral equation formulation of plate bending problems. *J. Engng Math.* **2**, 83–93 (1968).
3. M. Maiti and S. K. Chakrabarty, Integral equation solutions for simply supported polygonal plates. *Int. J. Engng Sci.* **12**, 793–806 (1974).
4. E. B. Hansen, Numerical solution of integro-differential and singular integral equations for plate bending problems. *J. Elast.* **6**, 39–56 (1976).
5. N. J. Altiero and D. L. Sikarskie, A boundary integral method applied to plates of arbitrary plan form. *Computers and Structures* **9**, 163–168 (1978).

6. M. L. Williams, Surface stress singularities resulting from various boundary conditions in angular corners of plate under bending. *Proc. 1st U.S. Nat. Cong. Appl. Mech.*, Chicago 325–329 (1951).
7. M. Stern, A boundary integral representation for stress intensity factors. *Recent Advances in Engineering Science*, Part II of Proc. of the 10th Anniv. Meeting of the Soc. of Engng Sci., Raleigh, N.C., November 1973; Vol. 9, pp. 125–132. Boston (1977).
8. S. A. Timoshsnko and S. Woinowsky-Kreiger. *Theory of Plates and Shells.* 2nd Edn. McGraw-Hill, New York (1959).
9. J. A. Wojtaszak, The calculation of maximum deflection, moment and shear for uniformly loaded rectangular plate with clamped edges, *J. Appl. Mech.* **4**, 173–176 (1937).
10. W. T. Moody, *Moments and Reactions for Rectangular Plates.* U.S. Department of Interior Bureau of Reclamation Engineering Monograph 27 (1960).